Marked Trees and Generating Functions with Odd Variables.

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September 25, 1998

Abstract

We show that the generating functions with one even and one odd variable can help us in counting planar rooted trees with an additional marking. Possible generalization for many odd variables are mentioned as well.

1.Trees and marked trees.

An equivalent way to discuss planar rooted trees is that of bracketings ([2,3]). We will use both ways throughout the article as if they are two languages to describe one and the same reality.

Let us remind the standard recursive definition of bracketings ([1,2]). First, x is a bracketing and a sequence $(B_1 \cdots B_k)$ is a bracketing provided $k > 1$ and each B_i is a bracketing. Thus $(x \cdot x)$ are bracketings for any number of x and for example

$$(((xxx)x)((xx)xxx)x)$$

is a bracketing. Traditionally the outmost brackets are ommitted so it would be written $xxxx$ insted of $(xxxx)$ but we do not follow this custom.

If $B = (B_1 \cdots B_k)$ is a bracketing then the corresponding tree $\tau(B)$ consists of a root and the trees $\tau(B_1)$, ..., $\tau(B_k)$ adjusted to it in that order

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from left to right, and if $B = x$ then the tree is the root itself. The tree will have no vertex of degree one and the number of endpoints equal to the number of x-es in the bracketing.

We will call an internal vertex any vertex other than an endpoint. It is clear that an internal vertex corresponds to a pair of coupled parentheses in the bracketing.

We say that a tree is vertex-marked (or v-marked) if one of its internal vertices is distinguished, or marked. When we talk of a v-marked bracketing, we mean that the corresponding tree is v-marked, or by that matter that a pair of coupled brackets is marked. Below are shown different v-marked bracketings that resulted from different marking of the same plain bracketing.

$$[((xxx)x)((xx)xxx)x] \quad ([(xxx)x]((xx)xxx)x) \quad (([xxx]x)((xx)xxx)x)$$
$$(((xxx)x)[(xx)xxx]x) \quad (((xxx)x)([xx]xxx)x)$$

For the following let $b(n)$ denote the total number of bracketings of a string of n letters x. These numbers are often called Schröder numbers, and the notation $s(n)$ is used in [2]. Let $v(n)$ be the number of v-marked bracketings of such a string of length n, or, what would be the same, the number of bracketings counted with the weight equal to the number of couples parentheses in a bracketing.

Another kind of marking that we would like to consider is related to the following definition.

We call a stem of a bracketing any maximal substring of the type $(\cdots($ in the bracketing. So for example there are two stems in the bracketing below that are underlined

$$\underline{(((}xxx)x)\underline{((}xx)xxx)x).$$

(In a sense these are left stems and the right ones could be defined similarly, but we will not use them in the text.)

Similarly we say that a tree or a bracketing is stem-marked (s-marked) if one of its stems is chosen, and we denote by $s(n)$ the total number of s-marked trees with n endpoints.

2.Odd variable.

We need only basis properties of the odd or grassmann variables and we consider the case of one odd variable first and return to the case of many odd variables later.

If ε is an odd variable then by definition $\varepsilon^2 = 0$ so

$$(a_1 + b_1\varepsilon)(a_2 + b_2\varepsilon) = a_1\, a_2 + (b_1\, a_2 + a_1\, b_2)\varepsilon,$$

where $a_i,\ b_i$ are numbers or functions (power series) in the usual, even variables.

This formula can be generalized to n multiples

$$\prod_{i=1}^{n}(a_i + b_i\varepsilon) = \prod_{i=1}^{n} a_i\ + (\sum_{k=1}^{n} a_1 \dots b_k \dots a_n)\varepsilon\,. \tag{1}$$

A function or power series $F(t,\varepsilon)$ of an even variable t and an odd one ε can by definition be written as

$$F(t,\varepsilon) = f_0(t) + f_1(t)\varepsilon$$

and on the other hand

$$f_0(t) = F(t,0),\ \ f_1(t) = \frac{\partial F}{\partial \varepsilon}(t,0).$$

If we substitute $t+\varepsilon$ in a function $f(t)$, we get

$$f(t+\varepsilon) = f(t) + f'(t)\varepsilon.$$

These are properties that will be of help in futher calculations.

3.Generating functions for marked trees.

Let us consider v-marking first.

As usual we define the generating function $b(t)$ for the numbers $\{b(n)\}$ of bracketings to be

$$b(t) = \sum_{n=1}^{\infty} b(n)t^n = t + \sum_{n=2}^{\infty} b(n)t^n,$$

(remember $b(1) = 1$). The standard way to calculate $\{b(n)\}$ is to find the formula for $b(t)$ ([s1,s2]) first. We would like to repeat the arguments in order to generalize them later on.

If a bracketing B is not x then $B = (B_1 \cdots B_k)$ and as we cut out the root of the tree or, the same but in an other language, remove the outmost parentheses we get an ordered k-tuple of bracketings $B_1, \ldots, B_k$ of lesser length. The generation function for the numbers of k-tuples is clearly $b(t)^k$ so we come to an equation

$$y - t = y^2 + y^3 + y^4 + \cdots = \frac{y^2}{1-y}$$

that would be satisfied for $y = b(t)$. This gives us a quadratic equation

$$2y^2 - (1+t)y + t = 0.$$

So we have as a result the formula for $b(t)$

$$b(t) = \frac{1}{4}(1 + t - \sqrt{1 - 6t + t^2}). \tag{2}$$

For the v-marked bracketings we put

$$v(t) = \sum_{n=2}^{\infty} v(n)t^n, \qquad F(t, \varepsilon) = b(t) + v(t)\varepsilon$$

(here $v(1) = 0$ as well).

Let us proceed by the same procedure of "cutting the root" as above. Here we have to consider two possibilities: either the root is marked or non-marked. In the first case we case we get k-tuple of non-marked trees, but in the second exactly one of the trees in the resulting k-tuple is marked. Taking the formula (1) into account we conclude that the generating function for the k-tuples would be $F^k + \varepsilon F^k$.

This gives us the following equation for $y = F(t, \varepsilon)$

$$y - t = (1+\varepsilon)(y^2 + y^3 + y^4 + \cdots) = (1+\varepsilon)\frac{y^2}{1-y},$$

which we can rewrite as the quadratic equation

$$(2+\varepsilon)y^2 - (1+t)y + t = 0.$$

Therefore we get a formula for F

$$F(t,\varepsilon) = \frac{2-\varepsilon}{8}(1 + t - \sqrt{1 - 6t + t^2 - 2t\varepsilon}). \tag{3}$$

We will see later on that this formula implies recurrent relations between the numbers $\{b(n)\}$, $\{v(n)\}$, but let us discuss the case of s-marking before.

Starting with the generating function

$$s(t) = \sum_{n=2}^{\infty} s(n)t^n,$$

(again $s(1) = 0$), we denote

$$G(t,\varepsilon) = b(t) + s(t)\varepsilon.$$

Similarly either the bracketing B is x or the can remove the root and there appear $B_1, \ldots, B_k$, but now there are more cases to consider:
a) the first stem in B was marked and $B_1 = x$,
b) the first stem in B was marked but $B_1 \neq x$,
c) the marked stem was not the first one but $B_1 = x$,
d) the marked stem was not the first one and $B_1 \neq x$.

In case (a) there is no marking in $B_1, \ldots, B_2$ but in cases (b), (c) and (d) the marking is still present, either in B_1 or in the one of $B_2, \ldots, B_k$.

Taking (1) into account we see that the cases (a) and (c) together are described by a generating function

$$(t + t\varepsilon)(G + G^2 + G^3 + \cdots),$$

and the cases (b), (d) together are counted by a generating function

$$(G - t)(G + G^2 + G^3 + \cdots),$$

hencefore we come to an equation

$$y - t = (y + t\varepsilon)(y + y^2 + y^3 + \cdots) = (y + t\varepsilon)\frac{y}{1-y}.$$

It gives us almost the same quadratic equation again

$$2y^2 - (1 + t - t\varepsilon)y + t = 0.$$

So we get a formula for G

$$G(t,\varepsilon) = \frac{1}{4}(1 + t - \sqrt{1 - 6t + t^2 - 2t(1+t)\varepsilon}), \tag{4}$$

we will use it to calculate a recurrent relation for the stem-numbers.

4.Recurrent relations.

The formula (2) for $b(t)$ implies the formula for $b'(t)$

$$b'(t) = \frac{1}{4}\left(1 + \frac{3-t}{\sqrt{1-6t+t^2}}\right).$$

Let us denote $R = \sqrt{1-6t+t^2}$. Hence

$$4\,b'(t) - 1 = \frac{3-t}{R}. \tag{5}$$

But $1 + t - 4b(t) = R$, so

$$(4\,b'(t) - 1)(1 - 6t + t^2) = (4\,b(t) - t - 1)(t - 3).$$

or

$$b'(t)(1 - 6t + t^2) - b(t)(t-3) + t - 1 = 0.$$

Now evaluating the coeffitients by t^n, $n > 1$ we get the recurrence ([s1]):

$$(n+1)\,b(n+1) - (6n-3)\,b(n) + (n-1)\,b(n-1) = 0.$$

Moving to v-marked trees we use (3) to get an expression for $v(t) = \partial F/\partial\varepsilon$,

$$v(t) = -\frac{1}{2}\,b(t) + \frac{t}{2R}.$$

Hence

$$2\,v(t) + b(t) = \frac{t}{R}. \tag{6}$$

The equations (5) and (6) imply that

$$(3-t)\,(2\,v(t) + b(t)) = t\,(4\,b'(t) - 1)$$

and so the recurrence ensues for $n > 1$:

$$6\,v(n) = 2\,v(n-1) + (4n-3)\,b(n) + b(n-1). \tag{7}$$

We can compute a recurence for the stem-numbers too. From (4)

$$s(t) = -\frac{t}{4} + \frac{t(1+t)}{4R}.$$

Hencefore, taking (5) into account we conclude that

$$(3-t)\left(s(t)+\frac{t}{4}\right)=(t+t^2)\left(b'(t)-\frac{1}{4}\right).$$

This results in the recurrence for $n > 2$

$$3\,s(n) = \ s(n-1) + n\,b(n) + (n-1)\,b(n-1). \tag{8}$$

Let us include also the actual values for these numbers for $n \leq 8$.

n	1	2	3	4	5	6	7	8	9	10
# plain bracketings	1	1	3	11	45	197	903	4279	20793	103049
# s-marked bracketings	0	1	4	19	96	501	2668	14407	78592	432073
# v-marked bracketings	0	1	5	26	138	743	40043	122468	122468	679757

5.Several odd variables.

We will the situation with sevaral odd variable appear only very briefly.

It is quite naturally to imagine that several odd variable would appear when we decided to count several kinds of marking simultaneously. We will consider in the folowing somewhat artificial example of this kind – v-marking and s-marking blended together.

Let us introduce first two odd variables ε_v and ε_s. The relations that ε_v and ε_s are to be chosen. Let them be of the type:

$$\varepsilon_v^2 = \varepsilon_s^2 = 0, \qquad \varepsilon_v\varepsilon_s = \alpha \cdot \varepsilon_s\varepsilon_v, \tag{9}$$

where the constant α are to be fixed later.

The compound generating function would be of the form

$$H(t,\varepsilon_v,\varepsilon_s) = b(t) + v(t)\,\varepsilon_v + s(t)\,\varepsilon_s + q(t)\,\varepsilon_s\varepsilon_v,$$

where $q(t)$ counts double marked bracketings.

Now we are at the point to define the sv-marked bracketing. The definition that is proposed below is made with the choice $\alpha = -1$ in mind. We fix this value for α futher on.

The sv-marked bracketing is the bracketing with a v-mark assigned to the left one of a chosen coupled parentheses and an s-mark assigned to a stem in the bracketing and it is counted with a sign -1 if v-mark is to the left of the s-mark and with a sign $+1$ otherwise.

Let us show how is works in examples:

$$\begin{array}{ll}
\underline{[((}xxx)x)((xx)xxx)x] & \cdots\ +1, \\
\underline{([(}xxx)x]((xx)xxx)x) & \cdots\ +1, \\
\underline{(((}xxx)x)[(xx)xxx)x] & \cdots\ +1, \\
[((xxx)x)\underline{((}xx)xxx)x] & \cdots\ -1, \\
([(xxx)x]\underline{((}xx)xxx)x) & \cdots\ -1, \\
(([xxx]x)\underline{((}xx)xxx)x) & \cdots\ -1.
\end{array}$$

Now we would like to write an equation for $H(t, \varepsilon_v, \varepsilon_v)$ in the same manner as before, that means we take off the root and consider the arising structure. We leave it to the reader to check that the result will be as follows:

$$y - t = \Big((1+\varepsilon_s)(ty + ty^2 + \cdots) + (y-t)y + (y-t)y^2 + \cdots\Big)\,(1+\varepsilon_v),$$

or

$$y - t = \Big((y + t\varepsilon_s)(y + y^2 + \cdots)\Big)\,(1+\varepsilon_v) = (y + t\varepsilon_s)y(1-y)^{-1}(1+\varepsilon_v).$$

This can be simplyfied

$$(y-t)(1-\varepsilon_v)(1-y) = y^2 + t\varepsilon_s y$$

and then made into a kind of quadratic equation

$$y(2-\varepsilon_v)y - (1 + t - t\varepsilon_v + t\varepsilon_s)y + y\varepsilon_v + t(1-\varepsilon_v) = 0.$$

It is not so clear how to solve this equation because we are now in a non-commutative domain.

The case when we use several odd variables subjected to noncommutative relations are much more difficult to handle.

References

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